Wigner equation for particles obeying an exclusion-inclusion principle

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We consider a nonlinear Schrödinger equation for particles obeying an exclusion-inclusion principle recently proposed by us and derive, within the Wigner formalism, a generalized quantum phase-space transport equation. We calculate the $\hbar$ expansion of this equation and study the quantum corrections, up to the $\hbar^2$ order, of the finite temperature Thomas-Fermi phase-space distribution. The effects and the quantitative relevance of the exclusion principle on the distribution function are considered for a system of fermionic particles in a harmonic confining potential. [S1063-651X(98)10502-0]

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I. INTRODUCTION

In recent years, great interest has been devoted to the use of quantum-mechanical phase-space distributions in many areas of physics including quantum optics, statistical physics, nonlinear physics, chaotic systems, quantum cosmology, collision processes [1–4]. In particular, in heavy-ion physics, several approaches based on collective and transport models (as, for instance, the hydrodynamics model, the intranuclear cascade model, the "hot spot" model and the molecular dynamics) are frequently employed to understand the relevant phenomenology [5–9].

The quantum many-body methods as the Hartree-Fock and the random phase approximations or the time dependent Hartree-Fock theory are powerful algorithms and allow us to explain many experimental results. However, the numerical implementation to treat realistic situations with these theories leads to excessively great computational times and efforts. For this reason and to obtain a clearer interpretation of the results in many-body physics there is a rising interest in the semiclassical approach of quantum mechanics both for the stationary and for the nonequilibrium phenomena.

The Wigner formulation of quantum statistical mechanics respects the rules of quantum mechanics such as Heisenberg's uncertainty principle and captures most features of the Boltzmann function, however, it does not contain quantum statistical or many-body effects due to the Pauli principle. Because the Wigner equation, which describes the quantum dynamics in the phase space, is deduced from the Schrödinger equation, to derive a quantum transport equation, which contains many-body or statistical effects, it is necessary to introduce these effects directly into the Schrödinger equation.

Recently, a new treatment to take into account many-body effects due to the Pauli principle has been considered [10]. In this reference, starting from a classical Markovian process, which obeys an exclusion-inclusion principle (EIP) acting in the coordinate space and using Nelson's stochastic quantization method, the following nonlinear Schrödinger equation (NLSE) has been derived:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla_{\mathbf{x}}^2\psi+V\psi+[W(\rho)+i\mathcal{W}(\rho,\mathbf{j})]\psi, \tag{1}$$

with

$$W(\rho)=\kappa\frac{\hbar^2}{4m}\left[\frac{\nabla_{\mathbf{x}}^2\rho}{1+\kappa\rho}+\frac{2-\kappa\rho}{2\rho}\frac{(\nabla_{\mathbf{x}}\rho)^2}{(1+\kappa\rho)^2}\right], \tag{2}$$

$$\mathcal{W}(\rho,\mathbf{j})=-\kappa\frac{\hbar}{2\rho}\nabla_{\mathbf{x}}\left(\frac{\rho\mathbf{j}}{1+\kappa\rho}\right), \tag{3}$$

and where the quantum current $\mathbf{j}$ is given by

$$\mathbf{j}=-\frac{i\hbar}{2m}(1+\kappa\rho)(\psi^*\nabla_{\mathbf{x}}\psi-\psi\nabla_{\mathbf{x}}\psi^*). \tag{4}$$

The potentials $W(\rho)$ and $\mathcal{W}(\rho,\mathbf{j})$ are due to the presence of the EIP and, being functions of the particle density $\rho(\mathbf{x},t)=|\psi(\mathbf{x},t)|^2$ and of the quantum current $\mathbf{j}(\mathbf{x},t)$, introduce complex nonlinearities in the Schrödinger equation. The real parameter κ (which has the dimensions of L^d, where d is the dimension of the space) characterizes the EIP: for $\kappa=0$ the EIP is absent and Eq. (1) becomes the standard linear Schrödinger equation; for $-1\leqslant\kappa\rho<0$ the EIP is reduced to an exclusion principle while for $\kappa\rho>0$ an inclusion principle holds [10].

Because the nonlinear potentials in the square bracket of Eq. (1) contain the EIP, the solution of Eq. (1) can reproduce an anyonlike particle behavior. The intermediate nature of bosons and fermions can be taken into account in the value of the parameter κ. In Ref. [10] it is shown that when the potential is parabolic, the real part of the nonlinear potential is reduced to $W(\rho)=-\kappa E_0\rho+(4\kappa\rho+3\kappa^2\rho^2)V(x)$. The first term of this potential appears in the Chern-Simons theories studying boson gasses with a δ-function pairwise repulsion or attraction [11]; the term in the brackets is similar to the one of the Eckhaus equation and describes a gas of bosons interacting with a two-body attractive and three-body repulsive δ-function interparticle potential [12]. The Eckhaus potential is often used in studies of superfluidity [13], superconductivity [14] and recently, of the Bose-Einstein condensation of trapped neutral atoms as, for instance, ^{7}Li, ^{23}Na, and ^{87}Rb [15–18].

The presence of the EIP does not change the usual interpretation of the function ρ as the particle density probability and of the vector $\mathbf{j}$ as the particle current probability, related to each other through the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{x}} \cdot \mathbf{j} = 0. \tag{5}$$

The NLSE (1) conserves in time the total density probability $\int \psi^* \psi d^3x$, the free equation $(V=0)$ is time and translation invariant and admits as solutions plane waves and solitons. Moreover, when Eq. (1) is considered in the momentum space, in the presence of a harmonic confining potential, the nonlinearities describe the fermion or boson or intermediate nature of the particles and then the quantity $|\psi(\mathbf{p})|^2$ defines Fermi-Dirac, Bose-Einstein, or intermediate quantum statistical distributions.

In this paper, we consider the above mentioned NLSE and, following the usual Wigner prescriptions of the phase-space quantum theory, we derive a quantum phase-space transport equation that describes the dynamical evolution of particles obeying an EIP. To our knowledge this is the first time that an EIP is introduced from the beginning into the Schrödinger equation to derive the phase-space dynamical equation. This allows us to generalize, in a natural mode, the collisionless dynamics of particles in the presence of an EIP.

The dynamical generalized Wigner equation and its stationary solutions are expanded in powers of $\hbar$. To illustrate quantitatively the relevance of the EIP introduced, we consider a pure fermionlike system in a harmonic confining potential and derive the semiclassical, extended finite-temperature Thomas-Fermi approximation up to the $\hbar^2$ order.

II. THE NONLINEAR WIGNER EQUATION

The phase-space representation of quantum mechanics was proposed by Wigner in 1932 [19], who studied the quantum corrections to the Boltzmann-Gibbs distribution with the introduction of the so-called Wigner function:

$$f(\mathbf{x},\mathbf{p},t) = N \int \frac{d^3y}{(2\pi\hbar)^3} e^{i\mathbf{p}\mathbf{y}/\hbar} \psi^*\left(\mathbf{x} - \frac{\mathbf{y}}{2}, t\right) \psi\left(\mathbf{x} + \frac{\mathbf{y}}{2}, t\right), \tag{6}$$

where N is the total number of particles in the system under consideration. The Wigner function $f(\mathbf{x},\mathbf{p},t)$, even though real, cannot be strictly interpreted as a distribution function in phase space, as the classical analog Boltzmann function, because it can have negative values. Nevertheless it can be considered as an auxiliary function useful for calculating thermodynamical averages, ground states of particles systems, time evolution of the density profiles, and multifragmentation in heavy-ion collisions [4–6,9,20].

In the framework of the Wigner formalism, the coordinate particle density $\rho(\mathbf{x},t)$ can be obtained by means of an integration over the momentum

$$\rho(\mathbf{x},t) = \frac{1}{N} \int d^3p\, f(\mathbf{x},\mathbf{p},t), \tag{7}$$

with the normalization taken as $\int \rho(\mathbf{x},t) d^3x = 1$ and the quantum particle current defined in Eq. (4) can be expressed as

$$\mathbf{j}(\mathbf{x},t) = (1+\kappa\rho)\mathbf{j}^{(0)}(\mathbf{x},t), \tag{8}$$

where $\mathbf{j}^{(0)}(\mathbf{x},t)$ is the particle current when the EIP is absent $(\kappa=0)$

$$\mathbf{j}^{(0)}(\mathbf{x},t) = \frac{1}{N} \int d^3p \frac{\mathbf{p}}{m} f(\mathbf{x},\mathbf{p},t). \tag{9}$$

With these ingredients we are able to derive the generalized quantum Wigner equation in phase space. Following standard procedures, which we have reported explicitly in the Appendix, the nonlinear Wigner equation in the presence of the EIP can be written as

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}}\right) f(\mathbf{x},\mathbf{p},t) = \frac{2}{\hbar}\left[\sin\left(\frac{1}{2}\hbar \triangle\right)[V(\mathbf{x}) + W(\rho)] + \cos\left(\frac{1}{2}\hbar\triangle\right)\mathcal{W}(\rho,\mathbf{j})\right] f(\mathbf{x},\mathbf{p},t), \tag{10}$$

where the triangle operator is defined as $\triangle = \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{p}}$ with $\nabla_{\mathbf{x}}$ acting only on the potentials $V(\mathbf{x})$, $W(\rho)$, $\mathcal{W}(\rho,\mathbf{j})$ and $\nabla_{\mathbf{p}}$ acting only on $f(\mathbf{x},\mathbf{p},t)$. In the right-hand side of Eq. (10) the sine and cosine functions must be interpreted as the implicit expressions of their formal expansions in powers of $\hbar$.

This equation describes the collisionless quantum dynamics of a nonrelativistic system of particles obeying the EIP, represents an extension of the standard linear Wigner equation [1–3], and reduces to it when the EIP is absent $(\kappa=0)$. We note that if the parameter κ has negative values, Eq. (10) describes the dynamics of particles that obey an exclusion principle in phase space. These are fermionlike particles and, in a first approximation, can be considered as fermions because the EIP acting in the coordinate space does not allow the double site occupancy in phase space. Because we limit ourselves to discussing the processes where the momentum changes slowly, the EIP in the coordinate space is consistent with our semiclassical approach. This requirement is equivalent to the condition that fast particles move in a slowly variable potential.

Equation (10) is highly nonlinear because the potentials $W(\rho)$ and $\mathcal{W}(\rho,\mathbf{j})$ contain the density ρ and the current $\mathbf{j}$ defined in terms of the Wigner distribution $f(\mathbf{x},\mathbf{p},t)$ through Eqs. (7) and (8). The solution of Eq. (10) is not trivial and can be integrated by means of self-consistent methods starting from a trial density function.

Integrating Eq. (10) in the momentum space, it is easy to verify that the continuity equation (5), where the density ρ is given by Eq. (7) and the particle current is defined by Eq. (8), is satisfied. Therefore the EIP modifies the particle current but conserves the total number of particles. The presence of the cosine term is a consequence of the different expression of the quantum particle current, which contains the corrective factor $1+\kappa\rho$. This modification gives rise to the imaginary potential $i\mathcal{W}$ in the Schrödinger equation (1). We remark that, in comparison to the standard linear equation, the field of applicability of Eq. (10) is the same as that of the standard one, but should be more appropriate to study physical effects where the many-body statistical effects are not negligible.

The expansion in powers of $\hbar$ of the standard linear Wigner equation has been considered in different physical applications and the quantum corrections, as a perturbation of the classical results, have been analyzed; discussions of the applicability and the convergence of the expansion into $\hbar$ powers are extensively considered in many papers [1,21–24].

III. EXPANSION IN POWERS OF $\hbar$

We wish now to expand the nonlinear Eq. (10) in terms of powers of $\hbar$ and to derive the quantum corrections up to $\hbar^2$ order of the Wigner distribution, outlining the relevant effects due to the presence of the potentials $W(\rho)$ and $\mathcal{W}(\rho,\mathbf{j})$. We expand formally the Wigner function $f=f(\mathbf{x},\mathbf{p},t)$:

$$f=f_0+\hbar f_1+\hbar^2 f_2+O(\hbar^3). \tag{11}$$

The particle density $\rho=\rho(\mathbf{x},t)$ and the particle current $\mathbf{j}=\mathbf{j}(\mathbf{x},t)$ can be expanded in $\hbar$ powers, being defined in terms of the Wigner function by means of Eqs. (7) and (8), therefore we can write the following expressions:

$$\rho=\rho_0+\hbar\rho_1+\hbar^2\rho_2+O(\hbar^3), \tag{12}$$

$$\mathbf{j}=\mathbf{j}_0+\hbar\mathbf{j}_1+\hbar^2\mathbf{j}_2+O(\hbar^3). \tag{13}$$

If we consider Eq. (2), the potential W can be written as

$$W=\hbar^2 M_0+O(\hbar^3), \tag{14}$$

where M_0 is given by

$$M_0=\frac{\kappa}{4m}\left[\frac{\nabla_{\mathbf{x}}^2\rho_0}{1+\kappa\rho_0}+\frac{2-\kappa\rho_0}{2\rho_0}\,\frac{(\nabla_{\mathbf{x}}\rho_0)^2}{(1+\kappa\rho_0)^2}\right]. \tag{15}$$

In the following, the potential $\mathcal{W}$ will be written as $\mathcal{W}=\hbar\mathcal{M}/2$, where the function $\mathcal{M}$ can be expanded as

$$\mathcal{M}=\mathcal{M}_0+\hbar\mathcal{M}_1+\hbar^2\mathcal{M}_2+O(\hbar^3), \tag{16}$$

where $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$ are given by the following expressions:

$$\mathcal{M}_0=-\kappa\frac{1}{\rho_0}\nabla_{\mathbf{x}}\cdot(\rho_0\mathbf{j}_0^{(0)}), \tag{17}$$

$$\mathcal{M}_1=-\kappa\frac{1}{\rho_0}\nabla_{\mathbf{x}}\cdot(\rho_0\mathbf{j}_1^{(0)}+\rho_1\mathbf{j}_0^{(0)})+\kappa\frac{\rho_1}{\rho_0^2}\nabla_{\mathbf{x}}\cdot(\rho_0\mathbf{j}_0^{(0)}), \tag{18}$$

$$\mathcal{M}_2=-\kappa\frac{1}{\rho_0}\nabla_{\mathbf{x}}\cdot(\rho_0\mathbf{j}_2^{(0)}+\rho_1\mathbf{j}_1^{(0)}+\rho_2\mathbf{j}_0^{(0)})+\kappa\frac{\rho_1}{\rho_0^2}\nabla_{\mathbf{x}}\cdot(\rho_0\mathbf{j}_1^{(0)}+\rho_1\mathbf{j}_0^{(0)})+\kappa\frac{\rho_2}{\rho_0^2}\nabla_{\mathbf{x}}\cdot(\rho_0\mathbf{j}_0^{(0)})-\kappa\frac{\rho_1^2}{\rho_0^3}\nabla_{\mathbf{x}}\cdot(\rho_0\mathbf{j}_0^{(0)}). \tag{19}$$

The functions ρ_n and $\mathbf{j}_n^{(0)}$ are, at $\hbar^n$ order, the particle density and the particle current, respectively, when the EIP is absent and are given by

$$\rho_n(\mathbf{x},t)=\frac{1}{N}\int d^3p\, f_n(\mathbf{x},\mathbf{p},t), \tag{20}$$

$$\mathbf{j}_n^{(0)}(\mathbf{x},t)=\frac{1}{N}\int d^3p\,\frac{\mathbf{p}}{m}f_n(\mathbf{x},\mathbf{p},t). \tag{21}$$

Equation (10) can be expanded in series of $\hbar$. Equalizing the terms with the same order of $\hbar$, we obtain a system of infinite time-dependent coupled equations for the functions $f_n(\mathbf{x},\mathbf{p},t)$. The first three equations of this system are

$$(\mathcal{D}-\mathcal{M}_0)f_0=0, \tag{22}$$

$$(\mathcal{D}-\mathcal{M}_0)f_1=\mathcal{M}_1 f_0, \tag{23}$$

$$(\mathcal{D}-\mathcal{M}_0)f_2=\mathcal{M}_1 f_1+\mathcal{M}_2 f_0+\triangle M_0 f_0-\tfrac{1}{8}\triangle^2\mathcal{M}_0 f_0-\tfrac{1}{24}\triangle^3 V f_0, \tag{24}$$

where the differential operator $\mathcal{D}$ is the total time derivative of the classical Boltzmann equation

$$\mathcal{D}=\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}-\nabla_{\mathbf{x}}V\cdot\nabla_{\mathbf{p}}. \tag{25}$$

If the initial distribution is known, the system of coupled equations [note that Eqs. (22)–(24) are the first three] allows one to obtain, univocally, the infinite functions $f_n(\mathbf{x},\mathbf{p},t)$ and therefore the function $f(\mathbf{x},\mathbf{p},t)$.

We note that also in the case where, as usual, the initial and boundary conditions are independent on $\hbar$, the expansion of the function f in powers of $\hbar$ contains all the powers of $\hbar$. Therefore, as will be seen later in detail, if we consider an expansion in powers of $\hbar$ of f, up to the second order, the term in $\hbar$ can be taken equal to zero. If we go above to the second order, also odd powers occur in the formal series of the Wigner function, contrary to the standard linear case where only even powers occur [1]. This difference is due to the presence of the EIP potential $\mathcal{W}$ in the cosine term of the generalized Wigner equation (10).

Integrating Eqs. (22)–(24) respect to the variable $\mathbf{p}$, it is possible to obtain the evolution equation for the first order momenta ρ_n of the function f_n defined by Eq. (20). The evolution equations of the densities ρ_n can also be obtained directly from Eq. (10), integrating with respect to $\mathbf{p}$ and then expanding $\rho(\mathbf{x},t)$ in powers of $\hbar$. It is easy to see that

$$\frac{\partial\rho_n}{\partial t}+\nabla_{\mathbf{x}}\cdot\mathbf{j}_n=0, \tag{26}$$

where $\mathbf{j}_n$ is the current at $\hbar^n$ order when the EIP is present and is given by

$$\mathbf{j}_n=\mathbf{j}_n^{(0)}+\kappa\sum_{m=0}^{n}\rho_m\mathbf{j}_{n-m}^{(0)}, \tag{27}$$

with $\mathbf{j}_n^{(0)}$ the current, given by Eq. (21), at $\hbar^n$ order when the EIP is absent $(\kappa=0)$.

Equation (26) implies the conservation of the particle number at any $\hbar$ order separately. This justifies the expansion of the Wigner function f to any $\hbar$ order to study the quantum effects as corrections of the classical distribution. The quantity ρ_0 obeys the continuity equation (26) with $n=0$, the current $\mathbf{j}_0$, given by Eq. (27) with $n=0$, can be written in the following simple expression:

$$\mathbf{j}_0=(1+\kappa\rho_0)\mathbf{j}_0^{(0)}. \tag{28}$$

Therefore, in the classical limit $\hbar\to 0$, one obtains the particle current $\mathbf{j}_0$ from the current $\mathbf{j}_0^{(0)}$ multiplied by the enhancing-inhibiting factor $1+\kappa\rho_0$ due to the EIP.

The presence of the EIP in the classical limit is a consequence of the fact that this principle has been introduced *ad hoc* in the particle kinetics, as one can note from Eq. (4) of the current paper. For this reason, Eq. (22) can be viewed as a generalized *classical* Boltzmann equation for particles embedded in the potential $\mathcal{M}_0$ and obeying an EIP.

IV. EXTENDED FINITE TEMPERATURE THOMAS-FERMI APPROXIMATION

Let us study explicitly the s stationary states where the particle current $\mathbf{j}$ and, as a consequence, the potential $\mathcal{W}=\hbar\mathcal{M}/2$ introduced by the EIP, vanish.

Equations (22)–(24) are reduced to the following equations:

$$\mathcal{D}_0 f_0=0, \tag{29}$$

$$\mathcal{D}_0 f_1=0, \tag{30}$$

$$\mathcal{D}_0 f_2=(\triangle M_0-\tfrac{1}{24}\triangle^3 V)f_0, \tag{31}$$

where we have defined the differential operator $\mathcal{D}_0$ as

$$\mathcal{D}_0=\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}-\nabla_{\mathbf{x}}V\cdot\nabla_{\mathbf{p}}. \tag{32}$$

The first two, Eqs. (29) and (30), imply only that the quantities $f_0(\mathbf{x},\mathbf{p})\equiv f_0(E)$ and $f_1(\mathbf{x},\mathbf{p})\equiv f_1(E)$ are arbitrary functions of the total energy $E=p^2/2m+V(x)$. When the initial and boundary conditions are independent of $\hbar$, the function f_1 must be taken identically equal to zero and the distribution f is given by $f=f_0+\hbar^2 f_2$; therefore, if the function f_0 is known, we can derive its quantum correction f_2 solving Eq. (31).

To describe the contribution of the EIP, we concentrate ourselves to a one-dimensional problem even if all the results could be extended to the three-dimensional case. Equation (31) can be easily integrated using, as independent variables, the coordinate x and the total energy E; its general solution is given by

$$\begin{aligned} f_2(x,E)=&\frac{1}{24m}\left[2V(x)\frac{d^2V(x)}{dx^2}-\left(\frac{dV(x)}{dx}\right)^2\right]\frac{d^3 f_0(E)}{dE^3}\\ &-\frac{1}{24m}\frac{d^2V(x)}{dx^2}\left[3\frac{d^2 f_0(E)}{dE^2}+2E\frac{d^3 f_0(E)}{dE^3}\right]\\ &+M_0(x)\frac{df_0(E)}{dE}. \end{aligned} \tag{33}$$

The function $f_0(E)$ must be chosen to approximate at best the exact stationary solution of the generalized Wigner equation. In the following we consider a fermionlike system ($\kappa=-1$) and the EIP becomes an exclusion principle (EP). For this system, we choose the function $f_0(E)=f_{\rm TF}(E)$ to be the Thomas-Fermi (TF) distribution

$$f_{\rm TF}(E)=\frac{1}{e^{\beta(E-\mu)}+1}, \tag{34}$$

where $\beta=1/k_BT$ and μ is the chemical potential fixed by the normalization. Let us remark that this choice is not univocal, any function of the classical Hamiltonian that approximates the exact stationary solution of the Wigner equation, can be appropriate. In literature, convolutions of the TF distribution (34) calculated with smoothing algorithms or Gaussian smoothing of the Wigner distribution function to improve the convergence of the perturbative expansion (11) are considered [1,22].

For a numerical evaluation of the quantum correction $\hbar^2 f_2(x,E)$ of the Thomas-Fermi distribution $f_0(E)$, we consider a system of fermions confined in a harmonic potential $V(x)=m\omega^2x^2/2$. With this potential, the first term in the square brackets of the Eq. (33) vanishes and the dependence on the coordinate of the function $f_2(x,E)$ is contained only in the last term of Eq. (33), which is due only to the EP. If we introduce the dimensionless variables $\epsilon=\beta E$ and $z=x/l$ where $l=\sqrt{2k_BT/m\omega^2}$, the $f(z,\epsilon)$ given by Eq. (11) can be written in the simple form

$$f(z,\epsilon)=f_0(\epsilon)-\hbar^2\omega^2\beta^2Q(\epsilon)-\hbar^2\beta M_0(z)f_0(\epsilon)[1-f_0(\epsilon)], \tag{35}$$

where

$$\begin{aligned} Q(\epsilon)=&\tfrac{1}{24}f_0(\epsilon)[3-2\epsilon-(9-14\epsilon)f_0(\epsilon)\\ &+6(1-4\epsilon)f_0(\epsilon)^2+12\epsilon f_0(\epsilon)^3]. \end{aligned} \tag{36}$$

Equation (35) is just the finite temperature equilibrium distribution of a system obeying an EP in a harmonic confining potential, with quantum corrections up to order $\hbar^2$. In the following, we indicate with $f_{\rm TF}=f_0$ the TF distribution (34), with $f_{\rm ETF}=f_0-\hbar^2\omega^2\beta^2Q$ the extended TF distribution with quantum corrections but without the EP and with $f_{\rm EP}$ the extended TF distribution with quantum corrections and EP.

V. NUMERICAL RESULTS AND DISCUSSION

To make clear the relevance of the quantum corrections of the unperturbed $f_{\rm TF}$, we consider explicitly the numerical

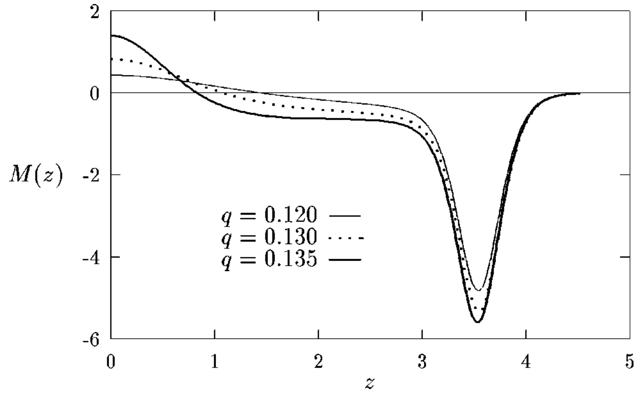

FIG. 1. The function M_0 (fm^{-1}) vs the dimensionless variable z at different values of q given in $\mathrm{fm}^{-1}(k_B=\hbar=1)$.

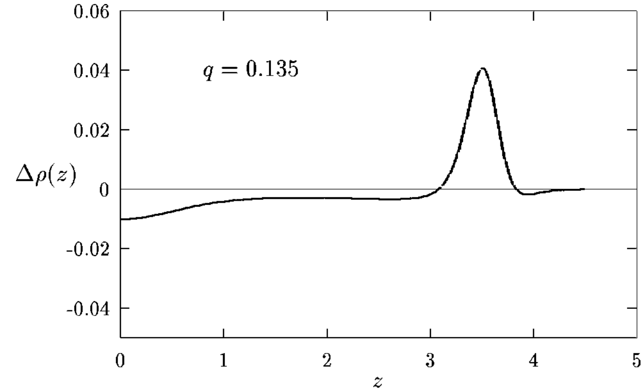

FIG. 2. The quantity $\Delta\rho(z)$ vs the dimensionless variable z for $q=0.135$ fm^{-1} and $T=0.8c$ fm^{-1}.

values of the different terms in Eq. (35). We choose a system with $N=60$ particles, $\omega=1$, and the units $k_B=\hbar=1$.

The potential $M_0=M_0(z)$ given by Eq. (15) is a crucial quantity because it indicates the presence of the EP introduced. It depends on the density ρ_0 at order $\hbar\rightarrow 0$, which can be calculated numerically, consistently with our approach, using Eq. (7) and the expression of the function $f_0=f_{\mathrm{TF}}$ introduced in Eq. (34). The direct use of the explicit expression of the wave function derived in [10] as a solution of the NLSE with harmonic confining potential is not consistent with the semiclassical extended finite temperature TF approximation. Furthermore, it is not possible from this wave function to compute analytically the Wigner function (6) and the relative density ρ. In Fig. 1 we report the function $M_0(z)$ for three different values of the parameter $q=\sqrt{2mk_BT}/N$.

As one can notice from Eq. (15), M_0 contains the terms $(d\rho_0/dz)^2$ and $d^2\rho_0/dz^2$; as a consequence the behavior of M_0 is very sensitive to the density in the region where the density changes its shape and curvature. When the density profile is constant, $M_0=0$. This is conceptually reasonable because we have introduced an EP in the coordinate space and this principle is more important in the region where there is a change of the density function. In Fig. 1, we may note that the function M_0 is repulsive for low z values and attractive for high z values (the curvature of M_0 changes where $M_0=0$).

To understand the physical action of M_0 or equivalently the EP effects in the stationary case, it is helpful to consider Fig. 2, where the difference $\Delta\rho(z)=(1/N)\int[f_{\mathrm{EP}}-f_{\mathrm{ETF}}]dp$ between the spatial density with and without the EP is plotted.

The presence of EP decreases the particle density in the region where the density is high and moves the particles towards the low density region, according to the confining harmonic potential, which is strongly repulsive in the high z region. This is the reason for the presence of the peak at high z values. Obviously, the number of particles is conserved so that $\int\Delta\rho(z)dz=0$.

In Fig. 3, we report the normalized distributions f_{TF}, f_{ETF} and f_{EP} versus the total energy E, for $q=0.135$ fm^{-1}, $T=0.8c$ fm^{-1}, and at the fixed value $z=0.2$ (in this region of variability of the parameter q the physical effect of the EP is more evident). For these values of z the potential introduced by the EP is repulsive and the distribution function f_{EP} is depleted with respect to the one (f_{TF} or f_{ETF}) without EP. Opposite effects are verified at higher z values where the EP potential is attractive.

As we have observed above, the profile of the f_{EP} is not symmetric in z and p as it is, on the opposite, in the absence of the EP (the distribution in this case depends only on the energy ϵ) because the EP holds in the coordinate space only. The f_{ETF} and the f_{EP} distributions show quantum oscillations, absent in the function f_{TF}; these can be identified with shell effects that vanish when averaged over the momentum.

VI. CONCLUSION

Within the Wigner scheme and prescriptions we have derived the quantum, nonlinear Eq. (10), having introduced a generalized Pauli principle into the nonlinear Schrödinger equation. Equation (10), the main result of this work, describes in phase space the quantum dynamics of a system of particles obeying to the EIP applied in the coordinate space, consistently with the semiclassical context (impulse slowly variable) in which we have developed our approach. Because Eq. (10) represents an extension of the linear Wigner equation in the presence of the EIP, the dynamical equation we have derived can be applied in all the physical fields where the standard equation holds (from quantum cosmology to condensed matter) in order to study physical problems of

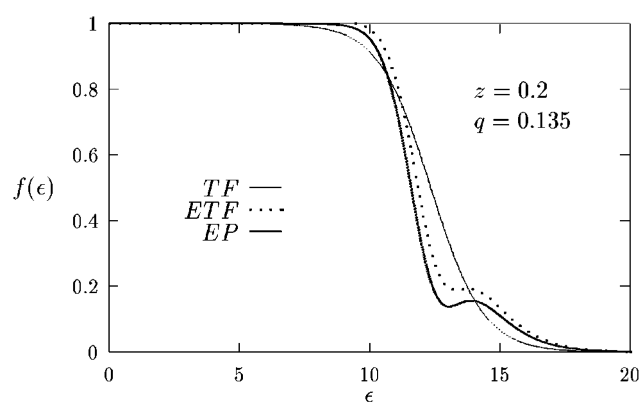

FIG. 3. The normalized distribution functions f_{TF}, f_{ETF}, f_{EP} vs the dimensionless energy ϵ at $z=0.2$ (dimensionless) and $q=0.135$ fm^{-1}.

systems in equilibrium or in nonequilibrium, where the many-body statistical effects are relevant.

We have shown that at any order of $\hbar$ the dynamical equation satisfies a continuity equation where the EIP is contained in the particle current expressed in general form by Eq. (27). Finally, we have considered the stationary states and the correction up to $\hbar^2$ of the finite temperature Thomas-Fermi distribution. Equation (35) gives the equilibrium distribution in phase space up to order $\hbar^2$ of a system of fermions obeying the EP in a harmonic confining potential.

APPENDIX

In this appendix we derive explicitly the nonlinear Wigner equation (10). It is convenient to introduce the potentials $U(\mathbf{x})$ and $\mathcal{W}(\mathbf{x})$, functions of $\mathbf{x}$, defined as

$$U(\mathbf{x})=V(\mathbf{x})+W(\rho),\qquad \mathcal{W}(\mathbf{x})=\mathcal{W}(\rho,\mathbf{j}). \tag{A1}$$

We use the operators $\vec{\nabla}_{\mathbf{x}}$ and $\vec{\nabla}_{\mathbf{p}}$ acting on the right side and the operators $\overleftarrow{\nabla}_{\mathbf{x}}$ and $\overleftarrow{\nabla}_{\mathbf{p}}$ acting on the left side. The starting point is, as usual, the nonlinear Schrödinger equation for the wave function $\psi(\mathbf{x},t)$:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x})=-\frac{\hbar^2}{2m}\vec{\nabla}^2_{\mathbf{x}}\psi(\mathbf{x})+U(\mathbf{x})\,\psi(\mathbf{x})+i\mathcal{W}(\mathbf{x})\,\psi(\mathbf{x}). \tag{A2}$$

The above NLSE for the wave function $\psi(\mathbf{x}-\mathbf{y}/2)$ multiplied by $\psi^*(\mathbf{x}+\mathbf{y}/2)$ can be written as

$$\begin{aligned}i\hbar\,\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\frac{\partial}{\partial t}\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)=&-\frac{\hbar^2}{2m}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\vec{\nabla}^2_{\mathbf{x}-\mathbf{y}/2}\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)+\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)U\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\\&+i\,\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\mathcal{W}\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right).\end{aligned} \tag{A3}$$

Analogously, the NLSE for $\psi^*(\mathbf{x}+\mathbf{y}/2)$ times $\psi(\mathbf{x}-\mathbf{y}/2)$ is given by

$$\begin{aligned}-i\hbar\,\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\frac{\partial}{\partial t}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)=&-\frac{\hbar^2}{2m}\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\vec{\nabla}^2_{\mathbf{x}+\mathbf{y}/2}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)+\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)U\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\\&-i\,\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\mathcal{W}\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right).\end{aligned} \tag{A4}$$

Subtracting Eq. (A4) from Eq. (A3), we obtain the following equation

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)=&-\frac{\hbar^2}{2m}\left[\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\vec{\nabla}^2_{\mathbf{x}-\mathbf{y}/2}\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\vec{\nabla}^2_{\mathbf{x}+\mathbf{y}/2}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right]+\left[U\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-U\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right]\\&\times\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)+i\left[\mathcal{W}\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\mathcal{W}\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right]\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right).\end{aligned} \tag{A5}$$

Taking into account the relation

$$\begin{aligned}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\vec{\nabla}^2_{\mathbf{x}-\mathbf{y}/2}\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\vec{\nabla}^2_{\mathbf{x}+\mathbf{y}/2}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)&=\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\vec{\nabla}^2_{\mathbf{x}}\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\vec{\nabla}^2_{\mathbf{x}}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\\&=\vec{\nabla}_{\mathbf{x}}\left[\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\vec{\nabla}_{\mathbf{x}}\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\vec{\nabla}_{\mathbf{x}}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right]\\&=-2\vec{\nabla}_{\mathbf{x}}\left[\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\vec{\nabla}_{\mathbf{y}}\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)+\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\vec{\nabla}_{\mathbf{y}}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right]\\&=-2\vec{\nabla}_{\mathbf{x}}\vec{\nabla}_{\mathbf{y}}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right),\end{aligned} \tag{A6}$$

and the identities

$$U\left(\mathbf{x}\pm\frac{\mathbf{y}}{2}\right)=U(\mathbf{x})\exp\left(\pm\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right),\quad \mathcal{W}\left(\mathbf{x}\pm\frac{\mathbf{y}}{2}\right)=\mathcal{W}(\mathbf{x})\exp\left(\pm\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right), \tag{A7}$$

Eq. (A5) can be written in the form

$$i\hbar\frac{\partial}{\partial t}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)=\frac{\hbar^2}{m}\vec{\nabla}_{\mathbf{x}}\vec{\nabla}_{\mathbf{y}}\left[\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right]-U(\mathbf{x})\left[\exp\left(\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right)-\exp\left(-\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right)\right]\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)$$
$$+i\mathcal{W}(\mathbf{x})\left[\exp\left(\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right)+\exp\left(-\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right)\right]\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right). \tag{A8}$$

Let us now multiply the first and second members of Eq. (A8) by $(2\pi\hbar)^{-3}\exp(i\mathbf{py}/\hbar)$ and integrating in the variable $\mathbf{y}$, we obtain the equation

$$\frac{\partial}{\partial t}\int\frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)$$
$$=-\frac{i\hbar}{m}\vec{\nabla}_{\mathbf{x}}\int\frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\vec{\nabla}_{\mathbf{y}}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)-\frac{1}{i\hbar}U(\mathbf{x})\int\frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\left[\exp\left(\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right)\right.$$
$$\left.-\exp\left(-\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right)\right]\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)+\frac{1}{\hbar}\mathcal{W}(\mathbf{x})\int\frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\left[\exp\left(\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right)\right]\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right). \tag{A9}$$

If we take into account the relation

$$\int\frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\vec{\nabla}_{\mathbf{y}}\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)=-\int\frac{d^3y}{(2\pi\hbar)^3}\vec{\nabla}_{\mathbf{y}}\left[\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\right]\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)$$
$$=-\int\frac{d^3y}{(2\pi\hbar)^3}\frac{i}{\hbar}\mathbf{p}\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)$$
$$=-\frac{i}{\hbar}\mathbf{p}\int\frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right), \tag{A10}$$

and the identity

$$A(\mathbf{x})\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\exp\left(\frac{1}{2}\mathbf{y}\overleftarrow{\nabla}_{\mathbf{x}}\right)B(\mathbf{x},\mathbf{y})=A(\mathbf{x})\sum_n\frac{1}{n!}\left[\mathbf{y}^n\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\right]\left(\frac{1}{2}\overleftarrow{\nabla}_{\mathbf{x}}\right)^nB(\mathbf{x},\mathbf{y})$$
$$=A(\mathbf{x})\sum_n\frac{1}{n!}\left[(-i\hbar\vec{\nabla}_{\mathbf{p}})^n\exp\left(\frac{i}{\hbar}\mathbf{py}\right)\right]\left(\frac{1}{2}\overleftarrow{\nabla}_{\mathbf{x}}\right)^nB(\mathbf{x},\mathbf{y})$$
$$=A(\mathbf{x})\exp\left(-\frac{i\hbar}{2}\overleftarrow{\nabla}_{\mathbf{x}}\vec{\nabla}_{\mathbf{p}}\right)\exp\left(\frac{i}{\hbar}\mathbf{py}\right)B(\mathbf{x},\mathbf{y}), \tag{A11}$$

Eq. (A9) can be written as

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int \frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{p}\mathbf{y}\right)\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\\
&\qquad = -\frac{\mathbf{p}}{m}\vec{\nabla}_{\mathbf{x}}\int \frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{p}\mathbf{y}\right)\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)+\frac{2}{\hbar}U(\mathbf{x})\frac{\exp\left(\frac{i\hbar}{2}\overleftarrow{\nabla}_{\mathbf{x}}\vec{\nabla}_{\mathbf{p}}\right)-\exp\left(-\frac{i\hbar}{2}\overleftarrow{\nabla}_{\mathbf{x}}\vec{\nabla}_{\mathbf{p}}\right)}{2i}\\
&\qquad\quad\times\int \frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{p}\mathbf{y}\right)\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)+\frac{2}{\hbar}\mathcal{W}(\mathbf{x})\frac{\exp\left(\frac{i\hbar}{2}\overleftarrow{\nabla}_{\mathbf{x}}\vec{\nabla}_{\mathbf{p}}\right)+\exp\left(-\frac{i\hbar}{2}\overleftarrow{\nabla}_{\mathbf{x}}\vec{\nabla}_{\mathbf{p}}\right)}{2}\\
&\qquad\quad\times\int \frac{d^3y}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathbf{p}\mathbf{y}\right)\psi^*\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right).
\end{aligned}
\tag{A12}
$$

Taking into account the Eulero expressions of the sine and cosine functions and remembering the definition of the Wigner function given by Eq. (6), Eq. (A12) turns into

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\vec{\nabla}_{\mathbf{x}}\right]f(t,\mathbf{x},\mathbf{p})=\frac{2}{\hbar}\left[U(\mathbf{x})\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{x}}\vec{\nabla}_{\mathbf{p}}\right)+\mathcal{W}(\mathbf{x})\cos\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{\mathbf{x}}\vec{\nabla}_{\mathbf{p}}\right)\right]f(t,\mathbf{x},\mathbf{p}), \tag{A13}$$

which is just the nonlinear Wigner equation of Eq. (10) with the potential $U(\mathbf{x})$ defined by Eq. (A1).

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